INEQUALITIES VIA $\varphi_{h,m}-$CONVEXITY

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Abstract. In this paper, we define $\varphi_{h,m}-$convex functions and prove some inequalities for this class.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard's inequality for convex functions, [2].

In [1], Toader defined $m-$convexity as the following.

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

In [4], Varošanec defined the following class of functions.

I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 2. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$,$\alpha \in (0, 1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y) \tag{1.2}$$

If inequality 1.2 is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$.

In [5], Sarıkaya et al. proved a variant of Hadamard inequality which holds for $h-$convex functions.

Theorem 1. *Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a, b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a)+f(b)]\int_0^1 h(\alpha)\,d\alpha. \tag{1.3}$$

In [3], Özdemir et al. defined $(h, m)-$convexity and obtained Hermite-Hadamard-type inequalities as following .

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Definition 3. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h,m)-$convex function, if f is non-negative and for all $x, y \in [0,b]$, $m \in [0,1]$ and $\alpha \in (0,1)$, we have*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y).$$

If the inequality is reversed, then f is said to be $(h,m)-$concave function on $[0,b]$.

Theorem 2. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(h,m)-$convex function with $m \in (0,1]$, $t \in [0,1]$. If $0 \leq a < b < \infty$ and $f \in L_1[ma,b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)\,dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right] \\ \leq\ & [f(a)+f(b)]\int_0^1 h(t)\,dt. \end{aligned}$$

Let us consider a function $\varphi : [a,b] \to [a,b]$ where $[a,b] \subset \mathbb{R}$. In [7], Youness defined the $\varphi-$convex functions as the following:

Definition 4. *A function $f : [a,b] \to \mathbb{R}$ is said to be $\varphi-$convex on $[a,b]$ if for every two points $x \in [a,b], y \in [a,b]$ and $t \in [0,1]$, the following inequality holds:*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + 1 - tf(\varphi(y)).$$

In [6], M.Z. Sarikaya defined φ_h-convex functions and obtained the following inequalities for this class.

Definition 5. *Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. We say that a function $f : I \to [0,\infty)$ is φ_h-convex if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) \tag{1.4}$$

for all $x, y \in I$ and $t \in (0,1)$. If inequality (1.4) is reversed, then f is said to be φ_h-concave.

Theorem 3. *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f : I \to [0,\infty)$ is Lebesgue integrable and φ_h-convex for continuous function $\varphi : [a,b] \to [a,b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(a)+\varphi(b)-x)dx \\ \leq\ & \left[f^2(\varphi(x)) + f^2(\varphi(y))\right]\int_0^1 h(t)h(1-t)\,dt + 2f(\varphi(x))\,f(\varphi(y))\int_0^1 h^2(t)dt. \end{aligned}$$

Theorem 4. *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f, g : I \to [0,\infty)$ is Lebesgue integrable and φ_h-convex for continuous function $\varphi : [a,b] \to [a,b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{\varphi(b)-\varphi(a)-}\int_{\varphi(a)}^{\varphi(b)} f(x)\,g(x)dx \\ \leq\ & M(a,b)\int_0^1 h^2(t)dt + N(a,b)\int_0^1 h(t)h(1-t)dt \end{aligned}$$

where

$$M(a,b) = f(\varphi(x))\,g(\varphi(x)) + f(\varphi(y))\,g(\varphi(y))$$

and

$$N(a,b) = f(\varphi(x))\, g(\varphi(y)) + f(\varphi(y))\, g(\varphi(x)).$$

The aim of this paper is to define a new class of convex function and then establish new Hermite-Hadamard-type inequalities.

2. MAIN RESULTS

In the begining we give a new definition $\varphi_{h,m}-$convex function.

I and J are intervals on $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 6. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : [0,b] \subseteq [0,\infty) \to \mathbb{R}$ is a $\varphi_{h,m}-$convex function, if f is non-negative and satisfies the inequality*

$$f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)) \tag{2.1}$$

for all $x, y \in [0,b], t \in (0,1)$.

If the inequality (2.1) is reversed, then f is said to be $\varphi_{h,m}-$concave function on $[0,b]$.

Obviously, if we choose $h(t) = t$ and $m = 1$ we have non-negative $\varphi-$convex functions. If we choose $m = 1$, then we have φ_h-convex functions. If we choose $m = 1$ and $\varphi(x) = x$ the two definitions $\varphi_{h,m}-$convex and $h-$convex functions become identical.

The following results were obtained for $\varphi_{h,m}-$convex functions.

Proposition 1. *If f, g are $\varphi_{h,m}-$convex functions and $\lambda > 0$, then $f + g$ and λf are $\varphi_{h,m}-$convex functions.*

Proof. From the definition of $\varphi_{h,m}-$convex functions we can write

$$f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)))$$

and

$$g(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)g(\varphi(x)) + mh(1-t)g(\varphi(y)))$$

for all $x, y \in [0,b], m \in (0,1]$ and $t \in [0,1]$. If we add the above inequalities we get

$$(f+g)(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)(f+g)(\varphi(x)) + mh(1-t)(f+g)(\varphi(y)).$$

And also we have

$$\lambda f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)\lambda f(\varphi(x)) + mh(1-t)\lambda f(\varphi(y)))$$

which completes the proof. □

Proposition 2. *Let $h_1, h_2 : (0,1) \to (0,\infty)$ be functions such that $h_2(t) \leq h_1(t)$ for all $t \in (0,1)$. If f is $\varphi_{h_2,m}-$convex on $[0,b]$, then for all $x, y \in [0,b]$ f is $\varphi_{h_1,m}-$convex on $[0,b]$.*

Proof. Since f is $\varphi_{h_2,m}-$convex on $[0,b]$, for all $x, y \in [0,b]$ and $t \in (0,1)$, we have

$$\begin{aligned} f(t\varphi(x) + m(1-t)\varphi(y)) &\leq h_2(t)f(\varphi(x)) + mh_2(1-t)f(\varphi(y))) \\ &\leq h_1(t)f(\varphi(x)) + mh_1(1-t)f(\varphi(y))) \end{aligned}$$

which completes the proof. □

Theorem 5. *Let f be $\varphi_{h,m}-$convex function. Then i) if φ is linear, then $f\circ\varphi$ is $(h-m)-$convex and ii) if f is increasing and φ is $m-$convex, then $f\circ\varphi$ is $(h-m)-$convex.*

Proof. i) From $\varphi_{h,m}-$convexity of f and linearity of φ, we have

$$\begin{aligned} f\circ\varphi\left[tx+m(1-t)y\right] &= f\left[\varphi\left(tx+m(1-t)y\right)\right] \\ &= f\left[t\varphi(x)+m(1-t)\varphi(y)\right] \\ &\leq h(t)f\circ\varphi(x)+mh\left(1-t\right)f\circ\varphi(y) \end{aligned}$$

which completes the proof for first case.

ii) From $m-$convexity of φ, we have

$$\varphi\left[tx+m(1-t)y\right]\leq t\varphi(x)+m(1-t)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f\circ\varphi\left[tx+m(1-t)y\right] &\leq f\left[t\varphi(x)+m(1-t)\varphi(y)\right] \\ &\leq h(t)f\circ\varphi(x)+mh\left(1-t\right)f\circ\varphi(y). \end{aligned}$$

This completes the proof for this case. □

Theorem 6. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$ and $f:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ be an $\varphi_{h,m}-$convex function with $m\in(0,1]$ and $t\in(0,1)$. Then for all $x,y\in[0,b]$, the function $g:[0,1]\to\mathbb{R}$, $g(t)=f(t\varphi(x)+m(1-t)\varphi(y))$ is $(h-m)-$convex on $[0,b]$.*

Proof. Since f is $\varphi_{h,m}-$convex function, for $x,y\in[0,b]$, $\lambda_1,\lambda_2\in(0,1)$ with $\lambda_1+\lambda_2=1$ and $t_1,t_2\in(0,1)$ we obtain

$$\begin{aligned} & g\left(\lambda_1 t_1+m\lambda_2 t_2\right) \\ =\; & f\left[\left(\lambda_1 t_1+m\lambda_2 t_2\right)\varphi(x)+m\left(1-\lambda_1 t_1-m\lambda_2 t_2\right)\varphi(y)\right] \\ =\; & f\left[\lambda_1\left(t_1\varphi(x)+m\left(1-t_1\right)\varphi(y)\right)+m\lambda_2\left(t_2\varphi(x)+m\left(1-t_2\right)\varphi(y)\right)\right] \\ \leq\; & h\left(\lambda_1\right)f\left(t_1\varphi(x)+m\left(1-t_1\right)\varphi(y)\right)+mh\left(\lambda_2\right)f\left(t_2\varphi(x)+m\left(1-t_2\right)\varphi(y)\right) \\ =\; & h\left(\lambda_1\right)g\left(t_1\right)+mh\left(\lambda_2\right)g\left(t_2\right) \end{aligned}$$

which shows the $(h-m)-$convexity of g. □

Theorem 7. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$ and $f:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ be a $\varphi_{h,m}-$convex function with $m\in(0,1]$ and $t\in(0,1)$. If $f\in L_1\left[\varphi(a),m\varphi(b)\right]$, $h\in L_1[0,1]$, one has the following inequality:*

$$\begin{aligned} & \frac{1}{m\varphi(y)-\varphi(x)}\int_{\varphi(x)}^{m\varphi(y)}f(u)f(\varphi(x)+m\varphi(y)-u)du \\ \leq\; & f^2\left(\varphi(x)\right)+m^2f^2\left(\varphi(y)\right)\int_0^1 h(t)h\left(1-t\right)dt+f\left(\varphi(x)\right)f\left(\varphi(y)\right)\left[m+1\right]\int_0^1 h^2(t)dt \end{aligned}$$

Proof. Since f is $\varphi_{h,m}-$convex function, $t\in[0,1]$ and $m\in(0,1]$, then

$$f(t\varphi(x)+m(1-t)\varphi(y))\leq h(t)f(\varphi(x))+mh(1-t)f(\varphi(y))$$

and

$$f((1-t)\varphi(x)+mt\varphi(y))\leq h(1-t)f(\varphi(x))+mh(t)f(\varphi(y))$$

for all $x,y\in[0,b]$.

By multiplying these inequalities and integrating on $[0,1]$ with respect to t, we obtain

$$
\begin{aligned}
& \int_0^1 f(t\varphi(x)+m(1-t)\varphi(y))f((1-t)\varphi(x)+mt\varphi(y))dt \\
\leq\ & f^2(\varphi(x))\int_0^1 h(t)h(1-t)dt + mf(\varphi(x))f(\varphi(y))\int_0^1 h^2(t)dt \\
& + mf(\varphi(x))f(\varphi(y))\int_0^1 h^2(1-t)dt + m^2 f^2(\varphi(y))\int_0^1 h(t)h(1-t)dt \\
=\ & \left[f^2(\varphi(x))+m^2 f^2(\varphi(y))\right]\int_0^1 h(t)h(1-t)dt + f(\varphi(x))f(\varphi(y))[m+1]\int_0^1 h^2(t)dt.
\end{aligned}
$$

If we change the variable $u=t\varphi(x)+m(1-t)\varphi(y)$, we obtain the inequality which is the required. □

Remark 1. *In Theorem 7, if we choose $m=1$ Theorem 7 reduces to Theorem 3.*

Theorem 8. *Under the assumptions of Theorem 7, we have the following inequality*

$$\frac{1}{m\varphi(y)-\varphi(x)}\int_{\varphi(x)}^{m\varphi(y)} f(u)\,du \leq [f(\varphi(x))+f(\varphi(y))]\int_0^1 h(t)dt.$$

Proof. By definition of $\varphi_{h,m}-$convex function we can write

$$f(t\varphi(x)+m(1-t)\varphi(y)) \leq h(t)f(\varphi(x))+mh(1-t)f(\varphi(y)).$$

If we integrate the above inequality on $[0,1]$ with respect to t and change the variable $u=t\varphi(x)+m(1-t)\varphi(y)$, we obtained the required inequality. □

Remark 2. *In Theorem 8, if we choose $m=1$ and $\varphi:[a,b]\to[a,b], \varphi(x)=x$, we obtained the inequality which is the right hand side of (1.3).*

Theorem 9. *Under the assumptions of Theorem 7, we have the following inequality*

$$
\begin{aligned}
& \frac{1}{m+1}\left[\frac{1}{\varphi(y)-m\varphi(x)}\int_{m\varphi(x)}^{\varphi(y)} f(u)\,du + \frac{1}{m\varphi(y)-\varphi(x)}\int_{\varphi(x)}^{m\varphi(y)} f(u)\,du\right] \\
\leq\ & [f(\varphi(x))+f(\varphi(y))]\int_0^1 h(t)dt
\end{aligned}
$$

for all $0\leq m\varphi(x)\leq\varphi(x)\leq m\varphi(y)<\varphi(y)<\infty$.

Proof. Since f is $\varphi_{h,m}-$convex function, we can write

$$
\begin{aligned}
f(t\varphi(x)+m(1-t)\varphi(y)) &\leq h(t)f(\varphi(x))+mh(1-t)f(\varphi(y)), \\
f((1-t)\varphi(x)+mt\varphi(y)) &\leq h(1-t)f(\varphi(x))+mh(t)f(\varphi(y)), \\
f(t\varphi(y)+m(1-t)\varphi(x)) &\leq h(t)f(\varphi(y))+mh(1-t)f(\varphi(x)),
\end{aligned}
$$

and

$$f((1-t)\varphi(y)+mt\varphi(x)) \leq h(1-t)f(\varphi(y))+mh(t)f(\varphi(x)).$$

By summing these inequalities and integrating on $[0,1]$ with respect to t, we obtain

$$\begin{aligned}
&\int_0^1 f(t\varphi(x)+m(1-t)\varphi(y))dt+\int_0^1 f((1-t)\varphi(x)+mt\varphi(y))dt \\
&+\int_0^1 f(t\varphi(y)+m(1-t)\varphi(x))dt+\int_0^1 f((1-t)\varphi(y)+mt\varphi(x))dt \\
\leq\ & \left[f\left(\varphi(x)\right)+f\left(\varphi(y)\right)\right](m+1)\left[\int_0^1 h(t)dt+\int_0^1 h(1-t)dt\right].
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\int_0^1 f(t\varphi(x)+m(1-t)\varphi(y))dt &= \int_0^1 f((1-t)\varphi(y)+mt\varphi(x))dt=\frac{1}{m\varphi\left(y\right)-\varphi\left(x\right)}\int_{\varphi(x)}^{m\varphi(y)} f\left(u\right)du, \\
\int_0^1 f(t\varphi(y)+m(1-t)\varphi(x))dt &= \int_0^1 f((1-t)\varphi(y)+mt\varphi(x))dt=\frac{1}{\varphi\left(y\right)-m\varphi\left(x\right)}\int_{m\varphi(x)}^{\varphi(y)} f\left(u\right)du
\end{aligned}$$

and

$$\int_0^1 h(t)dt=\int_0^1 h(1-t)dt.$$

If we write these equalities in the above inequality we obtain the required result. □

Remark 3. *In Theorem 9, if we choose $\varphi:[a,b]\to[a,b],\varphi(x)=x$ Theorem 9 reduces to Theorem 2..*

Theorem 10. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$ and $f,g:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ be $\varphi_{h,m}-$convex functions with $m\in(0,1]$. If f and g are Lebesque integrable, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{m\varphi\left(y\right)-\varphi\left(x\right)-}\int_{\varphi(x)}^{m\varphi(y)} f\left(u\right)g(u)du \\
\leq\ & M(a,b)\int_0^1 h^2(t)dt+mN(a,b)\int_0^1 h(t)h(1-t)dt
\end{aligned}$$

where

$$M(a,b)=f\left(\varphi(x)\right)g\left(\varphi(x)\right)+m^2 f\left(\varphi(y)\right)g\left(\varphi(y)\right)$$

and

$$N(a,b)=f\left(\varphi(x)\right)g\left(\varphi(y)\right)+f\left(\varphi(y)\right)g\left(\varphi(x)\right).$$

Proof. Since f and g are $\varphi_{h,m}-$convex functions, we can write

$$f(t\varphi(x)+m(1-t)\varphi(y))\leq h(t)f(\varphi(x))+mh(1-t)f(\varphi(y))$$

and

$$g(t\varphi(x)+m(1-t)\varphi(y))\leq h(t)g(\varphi(x))+mh(1-t)g(\varphi(y)).$$

If we multiply the above inequalities and integrate on $[0,1]$ with respect to t, we obtain

$$\begin{aligned}
&\int_0^1 f(t\varphi(x)+m(1-t)\varphi(y))g(t\varphi(x)+m(1-t)\varphi(y))dt \\
\leq\ & f\left(\varphi(x)\right)g\left(\varphi(x)\right)\int_0^1 h^2(t)dt+m^2 f\left(\varphi(y)\right)g\left(\varphi(y)\right)\int_0^1 h^2(1-t)dt \\
&+m\left[f\left(\varphi(x)\right)g\left(\varphi(y)\right)+f\left(\varphi(y)\right)g\left(\varphi(x)\right)\right]\int_0^1 h\left(t\right)h(1-t)dt.
\end{aligned}$$

If we change the variable $u=t\varphi(x)+m(1-t)\varphi(y)$, we obtain the inequality which is the required. □

Remark 4. *In Theorem 10, if we choose $m=1$ Theorem 10 reduces to Theorem 4.*

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